PHYS 320 ANALYTICAL MECHANICS

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Damped Oscillations (linear damping)

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$ with trial solution $x(t)=Ae^{qt}$

yields auxiliary equation

$$mq^2+cq+k=0 \quad \Rightarrow \quad q=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

∃ Three possibilities:

$$\left\{\begin{array}{ll} \mathrm{c}^2>4mk & \text{overdamped} \\ \mathrm{c}^2=4mk & \text{critically damped} \\ \mathrm{c}^2<4mk & \text{underdamped} \end{array}\right.$$

Damped Oscillations (linear damping)

auxiliary equation:

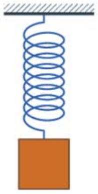
$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\gamma \pm \sqrt{\gamma^2 - \omega_o^2}$$

where $\gamma \equiv \frac{c}{2m} =$ damping parameter

(note that Taylor uses β)

and $\omega_o \equiv \sqrt{\frac{k}{m}} =$ natural frequency

Damped Oscillations (linear damping)



$c^2 < 4mk \quad (\gamma < \omega_o) \quad$ underdamped

yields actual oscillations!

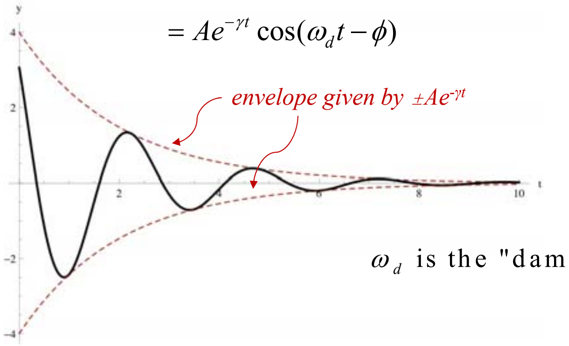
$$x(t) = C_+ e^{-\gamma t} e^{+i\sqrt{\omega_o^2 - \gamma^2}\, t} + C\_ e^{-\gamma t} e^{-i\sqrt{\omega_o^2 - \gamma^2}\, t}$$

$$= B_1 e^{-\gamma t} \cos(\omega_d t) + B_2 e^{-\gamma t} \sin(\omega_d t)$$

$$= A e^{-\gamma t} \cos(\omega_d t - \phi)$$

where

$$\begin{cases} \omega_d^2 \equiv \omega_o^2 - \gamma^2 \\ \phi \equiv \tan^{-1}(B_2 / B_1) \\ A^2 \equiv B_1^2 + B_2^2 \\ C_\pm \equiv (B_1 \mp iB_2)/2 \end{cases}$$



envelope given by $\pm Ae^{-\gamma t}$

ω_d is the "damped frequency"

(Note Taylor uses ω_1)

$\omega_o \equiv \sqrt{\frac{k}{m}}$

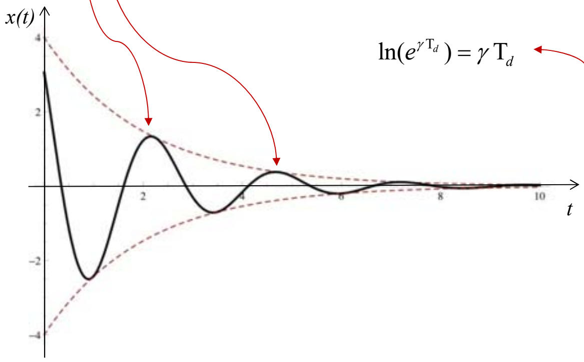
$\gamma \equiv \frac{c}{2m}$

Damped Oscillations (linear damping)

$$x(t) = C_+ e^{-\gamma t} e^{+i\omega_d t} + C_- e^{-\gamma t} e^{-i\omega_d t} = A e^{-\gamma t} \cos(\omega_d t - \phi)$$

Ratio of two successive maxima: $\frac{A\, e^{-\gamma t}}{A\, e^{-\gamma (t+T_d)}} = e^{\gamma T_d}$

decrement of motion



$\ln(e^{\gamma T_d}) = \gamma\, T_d$

logarithmic decrement of motion

$\boxed{\omega_d^2 \equiv \omega_o^2 - \gamma^2}$

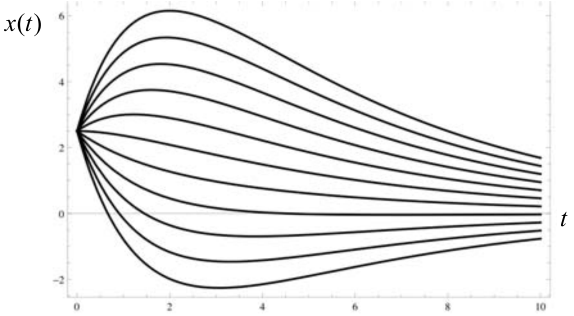
$\boxed{\omega_o \equiv \sqrt{\frac{k}{m}}}$

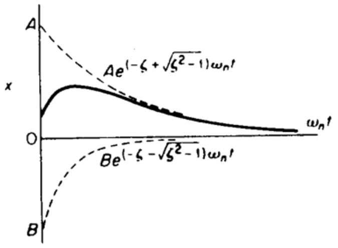
$\boxed{\gamma \equiv \frac{c}{2m}}$

Damped Oscillations (linear damping)

$$c^2 > 4mk \quad (\gamma > \omega_o) \quad \text{overdamped}$$

$$x(t) = A_1 e^{-\gamma t + \sqrt{\gamma^2 - \omega_o^2}\; t} + A_2 e^{-\gamma t - \sqrt{\gamma^2 - \omega_o^2}\; t}$$



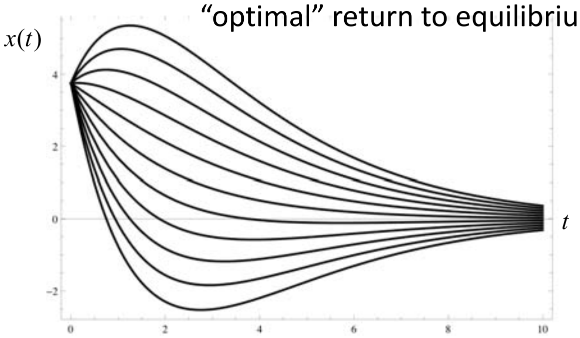


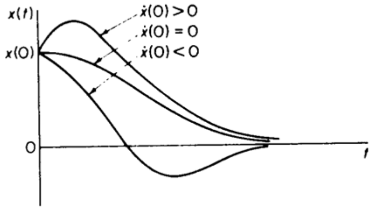
Damped Oscillations (linear damping)

$$c^2 = 4mk \quad (\gamma = \omega_o) \quad \text{critically damped}$$

$$x(t) = Ate^{-\gamma t} + Be^{-\gamma t}$$

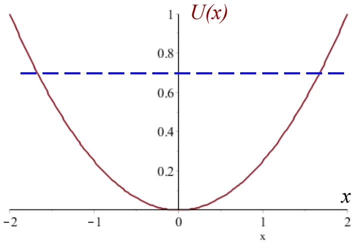
“optimal” return to equilibrium



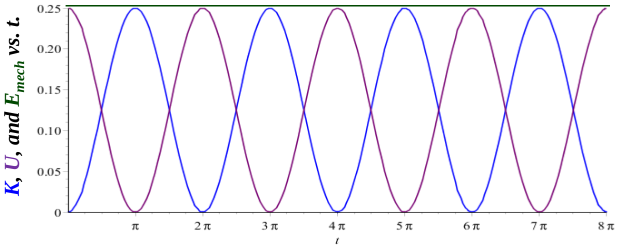


Simple Harmonic Oscillations (no damping)

Energy considerations $\quad x(t) = A\cos(\omega_o t - \phi)$



$$E_{mech}(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$



$$\omega_o \equiv \sqrt{\frac{k}{m}}$$

Differential Calculus: Del on vectors?

- The **del operator** is a **differential operator** that "*acts on*," rather than "multiplies" the function to its right.
- Acting on vectors, we have two options of interest:

$\vec{\nabla}\cdot\vec{V}$ **The Divergence**

$\vec{\nabla}\times\vec{V}$ **The Curl**

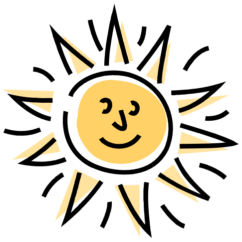
NOTE: *like the gradient, these are* ***COORDINATE SYSTEM DEPENDENT****!!*

Differential Calculus: The Divergence

- Gives feel for how much the field is *spreading out* or DIVERGING from the point in question!
- Often associate with *sources* or *sinks* of the field.
- Sort of a "*slope of the components*."
- Results in a scalar function!

$$\vec{\nabla}\cdot\vec{V}=\hat{i}\,\frac{\partial V_x}{\partial x}+\hat{j}\,\frac{\partial V_y}{\partial y}+\hat{k}\,\frac{\partial V_z}{\partial z}$$

The Divergence in Cartesian coords.



[COORDINATE SYSTEM DEPENDENT!!]

Differential Calculus: The Curl

- Gives feel for how much the vector field is *rotating* or CURLING about the point in question!
- Results in a vector function.

$$\vec{\nabla}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ B_x & B_y & B_z\end{vmatrix}$$

or

$$\vec{\nabla}\times\vec{B}=\hat{i}\,(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})+\hat{j}\,(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial y})+\hat{k}\,(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y})$$

The Curl in Cartesian coords.

[COORDINATE SYSTEM DEPENDENT!!]

A Theorem

"Curless" or irrotational vector functions

path (or line) integral

$$\left.\begin{aligned}&\text{i)}\quad \vec{\nabla}\times\vec{F}=0 \quad \forall \text{ space}\\ &\text{ii)}\quad \int_a^b \vec{F}\cdot d\vec{l} \quad \text{is path independent}\\ &\text{iii)}\quad \oint \vec{F}\cdot d\vec{l}=0 \quad \forall \text{ closed loops}\\ &\text{iv)}\quad \vec{F}=\vec{\nabla}V\end{aligned}\right\}\ \textbf{equivalent!}$$

Conservative forces are examples of such functions

Work, Force, and Potential Energy

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = -\Delta U = -(U_f - U_i)$$

path (or line) integral

where $\vec{F} = -\vec{\nabla} U$ and such forces are ***conservative***

Work and Kinetic Energy

$$W_{net} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} = \Delta K = (\mathrm{K}_f - K_i)$$

The Gradient

- Del acting on a scalar is called the **gradient**

$$\vec{\nabla} P(x,y,z) \equiv \hat{i}\frac{\partial P}{\partial x} + \hat{j}\frac{\partial P}{\partial y} + \hat{k}\frac{\partial P}{\partial z}$$

- $\vec{\nabla} P(x,y,z)$ points in the direction of maximum increase in the function $P(x,y,z)$

- $\left|\vec{\nabla} P(x,y,z)\right|$ gives the "slope" (rate of increase) along this maximal direction

Conservative Forces

- Conservative forces can be written as the *gradient* of some scalar function:

$$\vec{F} = -\vec{\nabla} U = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) U = -\hat{i}\frac{\partial U}{\partial x} - \hat{j}\frac{\partial U}{\partial y} - \hat{k}\frac{\partial U}{\partial z}$$

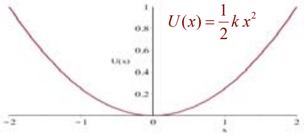
- One can show that this is equivalent to any of the following:

$$\vec{\nabla} \times \vec{F} = 0 \qquad\qquad \oint \vec{F} \cdot d\vec{r} = 0$$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \text{path independent}$$

Simple Harmonic Oscillations (no damping)

Energy considerations for a particle oscillating about a point of stable equilibrium.



Any potential well can be modeled as approximately parabolic for small enough oscillations

- Can do a Taylor series expansion about equilibrium position, x_o:

$$U(x) = \underbrace{U(x_o)}_{\text{const!}} + \underbrace{\left.\frac{dU(x)}{dx}\right|_{x_o}}_{=\,0} (x - x_o) + \underbrace{\frac{1}{2!}\left.\frac{d^2U(x)}{dx^2}\right|_{x_o} (x - x_o)^2}_{\text{looks like } \frac{1}{2} k\, x^2} + \underbrace{\ldots}_{\text{ignore small higher order terms!}}$$

- with $u = x - x_o$ and with $u < 1$, we can write

$$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m} \left.\frac{d^2U(x)}{dx^2}\right|_{\substack{u=0 \\ x=x_o}}}}$$

for small oscillations about equilibrium